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Some Problems in Knot Theory

A problem of a rather general nature is to find properties that it is necessary for a group G to have in order that it be a knot group (or a group of some analogous class, such as the class of link groups). One such property is that $G/G' \approx Z$ [hence $\Delta(1) = 1$]; another is that its defect is 1; and another is that its Alexander matrix class is "Hermitian," that is, $\mathbf{A}(t)$ is equivalent to the transpose of $\mathbf{A}(1/t)$. This is basically the property that I have called [1] "duality" in a knot group.

Problem 1. What group-theoretical property of G is responsible for the Hermitian character of the Alexander matrix class?

The Hermitian character of the elementary ideals (even for knots in arbitrary oriented 3-manifolds) is a consequence of the Blanchfield duality theorems [2]. These relate the homotopy chains of the complementary domain and therefore indicate a first step toward a solution of Problem 1.

Problem 2. Characterize, among the integral polynomials in μ variables, those which are link polynomials $\Delta(t_1, \ldots, t_\mu)$.

This is solved [3] for $\mu = 1$. For $\mu \geq 2$ several necessary conditions are known [1].

Problem 3. Is it true that every knot group can be represented as a free product $(F_n * F_n)_{F_{2n-1}}$ of the free group of rank n with itself amalgamated with respect to a free subgroup of rank $2n - 1$?

This is a conjecture of Neuwirth's, who has proved it for all alternating knots, along with all but one (9_{46}) of the nonalternating knots of fewer than 10 crossings [4]. The origin of the conjecture is, perhaps, the observation that the group of a torus knot can be so represented, with $n = 1$, $G = (x, y : x^a = y^b)$ for a torus knot of type (a, b). This is obtained by decomposing the complement of the torus knot into the part not inside the torus and the part not outside the torus, and applying van Kampen's theorem. If the conjecture were proved, various deep properties of G would follow, for example, the known fact that a knot group has no elements of finite order, and the conjectured fact that only torus knot groups have centers [4].

Problem 4. Is it possible to represent any pair of commuting elements of G by loops on a non-singular torus subset of the complementary domain?

If two elements commute, then naturally, representative loops lie together on a singular torus in the complementary domain. The problem is analogous to the Dehn lemma.

Problem 5. Is every knot group residually finite? That is, given any non-trivial element u of G, does there exist a homeomorphism θ of G into a finite group such that $\theta(u) = 1$?

Problem 6. Is the topological type of $S^3 - k$ determined by the group G together with its peripheral structure?

This has been solved by Neuwrith [4] for those knots k for which G' is finitely generated. Rapaport [5] has shown that G' cannot be finitely generated unless the leading coefficient of $\Delta(t)$ is ± 1.

Problem 7. Is the type of a knot k determined by the topological type of its complement $S^3 - k$?

This has been shown to be true for torus knots [6]. For links, the answer to the analogous question is in the negative [7]. For 2-spheres in 4-space the analogous question has been partly settled by H. Gluck [8].

Problem 8. If the group of a wild knot is finitely generated, is it a knot group?

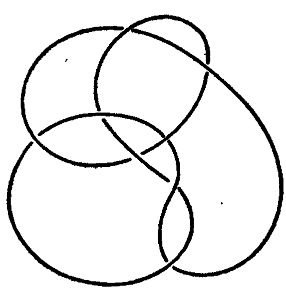
Of course, any knot group (= group of a tame knot) can be obtained as the group of a wild knot, simply by tying that kind of a knot in a good section of a wild knot whose group is Z.

Problem 9. Which pairs of knot types can be represented by the two components of a link of genus 1?

The genus of a link is defined in the same way as the genus of a knot, but note that it depends on the orientation of the components. If the orientation of one of the components is changed, the genus may change.

Problem 10. Do there exist non-invertible knots?

Among the 84 prime knots of 9 or fewer crossings, the only ones that are not obviously invertible [9] are 8_{17}, 9_{32}, 9_{33}, 9_{36}, 9_{44}, and 9_{45}. Although there is strong experimental evidence that 8_1 is not invertible, it has never been proved. To prove non-invertibility, it would be sufficient to prove the non-existence of any automorphism of G that maps a meridian into its inverse and a longitude into its inverse.



8_{17}

If 8_{17} were not equivalent to its inverse $\sigma(8_{17})$, then 8_{17} # 8_{17} and 8_{17} # $\sigma(8_{17})$ would be inequivalent knots.

Problem 11. Find conditions on a link of 2 components that must be fulfilled if the link is interchangeable.

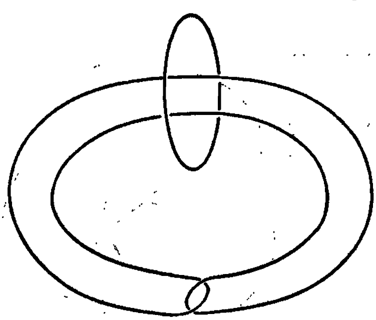
A link $K + L$ is *interchangeable* if there is an autohomeomorphism of space that maps K on L and L on K. An obvious condition is that K and L must be the same type of knot, but this is clearly insufficient.

Problem 12. Characterize, among the knot polynomials, those that are polynomials of alternating knots; of special alternating knots.

A knot is called *alternating* if it has a projection in which the undercrossings and overcrossings alternate around the knot. It is *special alternating* if it has an alternating projection in which one of the chessboard surfaces is orientable, that is, in which the Seifert circles are not nested. The polynomial of an alternating knot is an alternating polynomial in which no terms are skipped [10], that is, $\Delta(t) = \sum_{i=0}^{2h} (-1)^i c_i t^i$ and $c_i > 0$. Examination of the polynomials of the alternating knots of fewer than 12 crossings leads to the conjecture that $c_0 \leq c_1 \leq \ldots \leq c_h$, and that whenever $c_i = c_{i+1}$ for some $i < h$, then $c_i = c_{i+1} = \ldots = c_h$. The polynomial of a special alternating knot has the property [11] that c_0 and $\Delta(1)$ have the same sign.

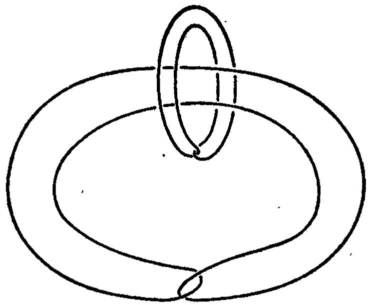
Problem 13. If $\Sigma - \Lambda$ is a covering of the complement $S^3 - L$, and K is a knot that is contractible in $S^3 - L$, then K is covered by a number of curves, indexed by the cosets of the subgroup to which the covering belongs. Can the linking numbers between these curves be computed from the group of the link $K + L$?

In 1937, Eilenberg [12] introduced an interesting concept that does not seem to have been investigated further. Call a link of two components 0-*linked* if the linking number is different from 0, and define, inductively, a link to be n-*linked* if every polyhedron in the complement of one of the components, say L_2, that contains the other component L_1 and on which some multiple of L_1 bounds, has on it a curve that is $(n - 1)$-linked with L_2. The link is 1-linked but not 0-linked, as was shown by Eilenberg.



Problem 14. Does there exist, for every positive integer n, a link that is n-linked but not $(n - 1)$-linked?

Good condidates for this are obtained from the link that is 1-linked but not 0-linked by doubling and redoubling, etc., for example, for $n = 2$



Problem 15. Is n-linking symmetric?

For $n = 1$, this follows from a characterization of Eilenberg's: L_1 and L_2 are 1-linked if the group of $L_1 + L_2$ cannot be mapped homeomorphically onto the free group of rank 2.

Problem 16. Does n-linking have anything to do with the identical vanishing of the Alexander polynomial $\Delta(x, y)$, or with the length of the chain of elementary ideals?

Problem 17. Are the arcs 1.1, 1.1*, 1.3 of "Some wild arcs . . ." [13] or the "remarkable simple closed curve" [14] invertible [7]? amphicheiral [7]?

Problem 18 (Ball). Do there exist uncountably many wild arcs that are locally tame except at one end point?

Problem 19. How is the local penetration index [15] related to the enclosure genus [15]? Can either of these numbers be calculated from the group and/or the local groups?

Problem 20. Do there exist wild simple closed curves whose maximal peripheral subgroups [16] are isomorphic to $Z \times Z$? to Z? to 1? Are these the only possibilities?

It would appear that the Bing sling [17] has no non-trivial peripheral element. It would also appear that a maximal peripheral subgroup of any wild knot must be either Z or 1.

Problem 21. Does there exist in 3-space or in 4-space a wild 2-cell with an interior point p such that every 2-cell subset that has p on its boundary is tame?

This seems to be a natural generalization of the wild arc that is the sum of two tame arcs. [13]. In 4-space, a good candidate is obtained by rotating about a plane an arc in half 3-space in which a sequence of knots is tied converging to the end point that is in the plane.

Problem 22. Does there exist in 3-space or in 4-space a non-invertible 2-cell? A non-amphicheiral 2-cell?

In our paper in these proceedings, Orville Harrold and I showed that in 3-space there is a non-amphicheiral arc, and modulo an affirmative solution to Problem 10, an arc that is non-invertible (and not for a trivial reason). The 2-cell asked for in this problem should be locally tame at every point except one interior point.

Problem 23. Is the granny knot a slice knot [7]?

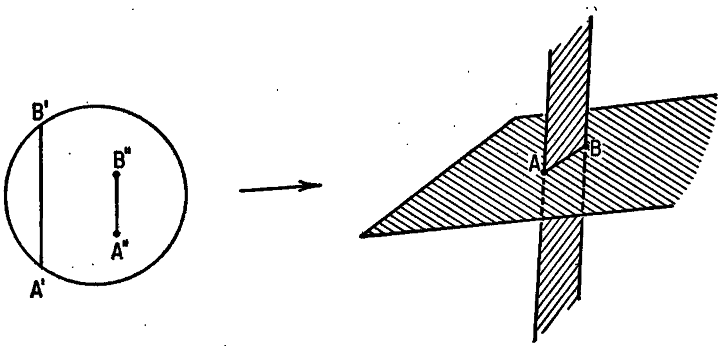
The expected negative answer to this question would establish the insufficiency of the polynomial condition [7] $\Delta(t) = F(t)F(1/t)$ for a knot to be a slice knot.

Let h^* denote the minimum of the genera of the locally flat orientable surfaces in half 4-space H^4 bounded by a knot or link L. Clearly $h^* \leq h$, the genus of L, but equality need not hold.

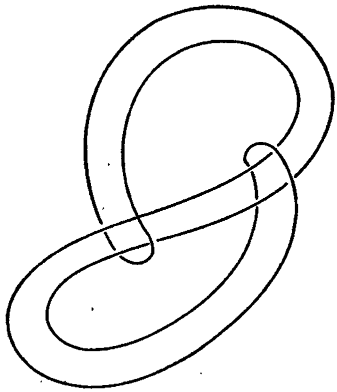
Problem 24. Find conditions that the group of L must satisfy in order that $h^* \leq$ *a* given positive integer N.

A knot is a slice knot iff $h^* = 0$, so the polynomial condition $\Delta(t) = F(t)F(1/t)$ is a partial answer to this problem.

Let us call a singular disk in R^3 a *ribbon* if the singularities are all of the following type:



and let us call a knot a *ribbon knot* if it is the boundary of a ribbon. Clearly any ribbon knot is a slice knot. (Deform a neighborhood of the line $A''B''$ into H^4.) Simple examples of slice knots are ribbon knots. For example, the square knot is a ribbon knot.

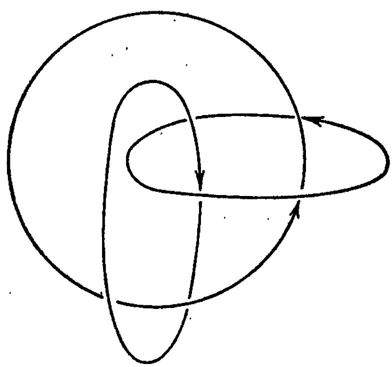


Problem 25. Is every slice knot a ribbon knot?

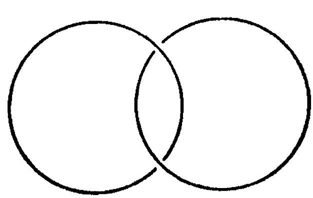
Let L be a link of μ components and consider the following four properties:

(1) L is a cross section of a union of μ locally flat 2-spheres in 4-space R^4;
(2) L bounds μ locally flat 2-cells in half 4-space H^4;
(3) L is a cross section of a single locally flat 2-sphere in R^4;
(4) L bounds a surface of genus 0 in H^4.

It is easy to see that (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4);



satisfies (3) but not (2), and



satisfies (4) but not (3). If $\mu \leq 2$, (2) $\Leftrightarrow$ (3), and, if $\mu = 1$, (3) $\Leftrightarrow$ (4).

I shall call L a *slice link* if it has property (3), and a *slice link in the strong sense* if it has property (1) = (2).

Problem 26. Find a necessary condition for L to be a slice link; a slice link in the strong sense.

Problem 27 (Kosinski). Can one find in S^4 a locally flat $S^2 \times S^1$ neither of whose (closed) complementary domains is what you would expect ($S^2 \times E^3$ or $E^3 \times S^1$)?

This is a natural generalization of a very powerful theorem of Alexander's [18]: if $S^1 \times S^1$ is tame in S^3, then at least one of the closed complementary domains is a solid torus.

If one has a locally flat 2-sphere (in 4-space) represented by cross sections, then it is easy to deform it so that all the minima are below all the maxima. It then follows that the group of the 2-sphere is a homomorph of the group of any of its cross-sectional links, just so long as one cuts between the maxima and minima.

Problem 28. If a finitely presented group is a homomorph of a knot group and is infinite cyclic over its commutator subgroup, it is the group of a locally flat 2-sphere in R^4.

Since there are locally flat 2-spheres in R^4 whose groups are not knot groups (Examples 10, 11 and 12), the following question immediately comes to mind.

Problem 29. Does there exist a locally flat 3-sphere in 5-space whose group is not the group of any locally flat 2-sphere in 4-space?

Problem 30. If the group of a locally flat 2-sphere in R^4 is $\approx Z$ is the 2-sphere trivial?

Problem 31. Is there more than one type of locally flat projective plane in R^4 whose group is $\approx Z_2$?

It follows from the Alexander duality theorem that if G is the group of a surface in R^4, then G/G' is Z or Z_2, depending on whether the surface is orientable or not. It is not difficult to construct a locally flat projective plane for which $G \approx Z_2$.

Problem 32. Can the group of a locally flat projective plane in R^4 be a finite group other than Z_2?

Problem 33. Does there exist a locally flat 2-sphere whose group has an element of even order?

Problem 34. Does there exist a locally flat 2-sphere whose group is $Z \times D$ where D is the dodecahedral group?

The origin of this question is an example constructed by Barry Mazur of a locally flat 2-sphere in a homotopy 4-sphere whose group is precisely $Z \times D$. A negative answer to Problem 34 would therefore imply that the 4-dimensional Poincaré conjecture is false.

Problem 35. Does there exist a non-amphicheiral locally flat 2-sphere?

Kinoshita observed that if $\Delta(t)$ is not a reciprocal polynomial, then there can be no automorphism of G that maps the generators x_i into their inverses, and consequently, a 2-sphere with such a polynomial is non-invertible. The same argument shows that such a 2-sphere is not +amphicheiral. (Examples 10 and 11 have non-reciprocal polynomials.) Thus Problem 35 asks whether there are any locally flat 2-spheres that are not −amphicheiral, that is, not transformable into their reflected inverses by an orientation-preserving homeomorphism. The fact that the simplest way to construct a slice knot is to compose a knot with its reflected inverse adds point to the problem.

Problem 36. Find an algorithm for calculating the second homotopy group of the complementary domain of a locally flat 2-sphere (regarded as an operator group, with the group of the 2-sphere as the group of operators).

Problem 37. Do there exist aspherical locally flat 2-spheres?

There is a class of 2-spheres in R^4 whose asphericity is known [19], but they are not locally flat.

Problem 38. Construct Brunnian [20] systems of 2-spheres in R^4.

This problem, which is probably rather easy, asks whether one can find μ disjoint locally flat 2-spheres such that any k of them are completely splittable, but no $k + 1$ of them are splittable at all.

Problem 39. (a) Which slice knots are cross sections of trivial 2-spheres? (b) Which slice links are cross sections of trivial 2-spheres? (c) Which slice links in the strong sense are cross sections of a trivial union of trivial 2-spheres?

I suspect that the stevedore's knot is not a cross section of any trivial 2-sphere. My reason for thinking this is that every attempt to destroy the Alexander ideal $\mathcal{E}_1$ of the stevedore's knot by extending it to a locally flat 2-sphere in R^4 seems to fail. In connection with parts (b) and (c) of Problem 39, compare examples 13 and 14.

Problem 40. How many ends [21] does the group of a locally flat 2-sphere have?

This generalizes a question settled by Papakyriakopoulos for knot groups. A knot group has two ends or one end, depending on whether the knot is trivial or not. Problem 40 asks merely whether there is any locally flat 2-sphere whose group has an infinite number of ends.

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